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Technical Notes

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Alternative Solution of the Kernel Function in Subsonic Unsteady Lifting Surface Theory

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Introduction

THE calculation of subsonic, compressible aerodynamic flows over harmonically oscillating lifting surfaces is an interesting problem because of the subtle difficulties in the solution process and the widespread application of the results. The governing equation of this complex flow was originally formulated by Küssner in 1940.[1] Subsequently, many efforts have been made to solve this equation. The most generally used scheme is the doublet lattice method (DLM).[2] Later, to expedite the solution process, the doublet point method (DPM)[3] was developed with simplifications in the doublet representation over the original doublet lattice method.

Both methods represent the lifting surface as a grid of trapezoidal-shaped boxes. In the DLM, a line of pressure doublets of constant strength is located at the one-quarter chord of each box, thereby representing the unsteady lift per unit span of the doublet line. In the DPM, a single doublet is placed at the one-quarter chord, on the midspan of each box, thus representing the net lift as a point force on the box. In both cases, the upwash at the control points of all of the boxes of the grid is computed using the kernel function formulated by Kussner.[1] Recently, a hybrid doublet lattice/doublet point method,[4] which utilizes the advantages of each solution scheme within the framework of a hybrid formulation, has been proposed.

Although the doublet point method is fairly simple in principle, there are several mathematical subtleties associated with derivation of the kernel function. The doublet distribution is first integrated in the chordwise direction, thereby constructing the doublet equivalent of horseshoe vortices. The resulting kernel function involves integrals that cannot be evaluated directly.

In a NACA report written in the late 1950s, Watkins et al.[5] discuss the derivation of the kernel function in great detail and offer an accurate approximate method for the solution of a modified form of the kernel function. Later, other researchers[6–10] developed alternative methods for solving the problem, either by modifying the work of Watkins et al. or by developing their own novel approach. To this end, exact solutions for evaluating the these integrals were derived in 1992 by Bismarck-Nasr[11] based on a differential equation approach. Houbolt[12] developed an averaging scheme in which the kernel function is modified in such a fashion that upwash velocities are averaged over equally spaced chordwise intervals, thereby alleviating the singular nature of the integral. Note, however, that many of these techniques involve numerical quadrature or special functions that are not readily calculated in numerical applications. Many of these techniques are at use in currently available industrial production codes.

The purpose of this Note is to outline another possible approach to the approximation of the kernel function. To this end, the kernel function has been approximated with a portion of the integrand in the kernel replaced with a simple curve fit. This curve fit employs functions that render the integrand readily integrable. The accuracy of this approximation could be questioned, but as shown in example calculations, the presented method compares well to the generally accepted asymptotic results of Ueda.[3]

Because of the closed-form nature of this solution, there is no issue of convergence, and therefore the method offers a wide range of applicability. In addition, the kernel function representation is fairly simple in form and is computationally efficient. However, it must be noted that the reduced integral form possesses a numerical divergence and can become less accurate for very small values of the argument. In most aerodynamic calculations, this should not present a problem because this argument range is out of the region needed for practical application.

Alternative Integration

The integral equation relating lift to downwash for a planer wing undergoing harmonic vibrations in subsonic flows can be written as a function of Mach number M, reduced frequency k, and coordinate positions $(x, y, x_0, y_0, \xi, \eta)$, where (ξ, η) are influence point positions, (x, y) are collocation point positions, and (x_0, y_0) are differential lengths defined between the influence points and the collocation points[1,2]

$$\frac{w(x, y)}{U} = \frac{1}{8\pi} \int\!\!\int \frac{\Delta p(\xi, \eta)}{q} K[M, k, x_0, y_0] \, \mathrm{d}A \tag{1}$$

where the kernel function K is

$$K[M, k, x_0, y_0] = e^{ikx_0} \frac{M e^{ikX}}{R\sqrt{X^2 + r^2}} + B \tag{2}$$

$$B(k, r, x) = \int_{-\infty}^{X} \frac{e^{ikv}}{(r^2 + v^2)^{\frac{3}{2}}} \, \mathrm{d}v \tag{3}$$

and

$$r^2 = (y - \eta)^2, \qquad X = \frac{x_0 - MR}{\beta^2}$$

$$R = \sqrt{x_0^2 + \beta^2 r^2}, \qquad \beta^2 = 1 - M^2$$

Specifically, this Note addresses the integral of Eq. (3), where the arguments k and r are nonnegative. The singularity in this integral can be easily identified with a simple change of variable $u = v/r$ and can be written[4]

$$B(k, r, X) = \frac{1}{r^2} \int_{-\infty}^{\frac{X}{r}} \frac{e^{ikru}}{(1 + u^2)^{\frac{3}{2}}} \, \mathrm{d}u \tag{4}$$

In this form it can be seen that as $r \to 0$, $B(k, r, X)$ becomes singular and has the limiting form

$$B(k, r \to 0, X) \Rightarrow \frac{2}{r^2} \tag{5}$$

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Separating real and imaginary parts, Eq. (4) can be written

$$B(k, r, X) = \frac{1}{r^2}\int_{-|\frac{X}{r}|}^{\infty} \cos(kru)\mathcal{F}(u)\,\mathrm{d}u + \frac{i}{r^2}\int_{-|\frac{X}{r}|}^{\infty} \sin(kru)\mathcal{F}(u)\,\mathrm{d}u \tag{6}$$

where the function $\mathcal{F}(u)$ is defined by

$$\mathcal{F}(u) = \frac{1}{(1+u^2)^{\frac{3}{2}}} \tag{7}$$

Equation (6) can be rewritten as

$$B_r(k, r, X) = \frac{1}{r^2}\int_0^{|\frac{X}{r}|} \cos(kru)\mathcal{F}(u)\,\mathrm{d}u + \frac{1}{r^2}\int_0^{\infty} \cos(kru)\mathcal{F}(u)\,\mathrm{d}u \tag{8}$$

$$B_i(k, r, X) = \frac{1}{r^2}\int_0^{|\frac{X}{r}|} \sin(kru)\mathcal{F}(u)\,\mathrm{d}u - \frac{1}{r^2}\int_0^{\infty} \sin(kru)\mathcal{F}(u)\,\mathrm{d}u \tag{9}$$

where the sign of the first integral in Eq. (8) will change if X is negative. Using a simple curve fitting technique, $\mathcal{F}(u)$ can be approximated as

$$\mathcal{F}(u) \cong \begin{cases} \mathcal{F}_1(u) & \text{if } u \le 1.5 \\ \mathcal{F}_2(u) & \text{if } u > 1.5 \end{cases} \tag{10}$$

where $\mathcal{F}_1(u)$ and $\mathcal{F}_2(u)$ are defined as

$$\mathcal{F}_1(u) = 1.0 - 0.6584u^2 + 0.1288u^4 + \frac{0.15}{2}\left(\cos\frac{2\pi u}{1.5} - 1.0\right) \tag{11}$$

$$\mathcal{F}_2(u) = u^{-3} - u^{-5} + (1/10)u^{-7} \tag{12}$$

with a maximum error of less than half a percent, normalized by the maximum of $\mathcal{F}(u)$, over the domain. Using this approximation, B_r and B_i, Eqs. (8) and (9) can be integrated, piecewise, term by term, over the domain in analytical form. Alternatively, the second integral on the right-hand side of Eqs. (8) and (9) can be integrated exactly, without the approximate form for $\mathcal{F}(u)$; however, this results in modified Struve functions.

For a specified reduced frequency k, $B_r(k, r, X)$ and $B_i(k, r, X)$ describe curvilinear surfaces parameterized by r and X. Therefore, for a given k and r, these equations describe a curve in space as a function of X. Figure 1 shows a comparison between the present method and the series expansion method[13] for the calculation of $B_r(1, 1, X)$ and $B_i(1, 1, X)$. Ten terms were retained in the series expansion. The reduced frequency k and the spanwise variable r are both unity. The agreement between the methods is excellent considering the simplicity of the curve fit used in the present analysis. Note that both techniques model the detailed shape of the curve well. Figure 2 shows another comparison between the methods, holding the reduced frequency constant, and reducing the spanwise variable by an order of magnitude such that $r = 0.1$. Again, the agreement between the methods is excellent. The singular nature of the integral is evident for small r as $X \to 0$.

Figure 3 shows a comparison between the methods for calculating $B_i(4, 4, X)$, where $k = 4$ and $r = 4$. As shown in the figure, it takes over a hundred terms for the series expansion to converge in this moderate case. The present method requires no additional computational cost for larger values of k and r.

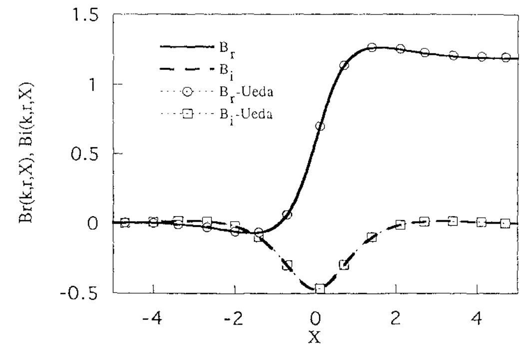

Fig. 1 Comparison between the present method and the series expansion method for k = 1.0 and r = 1.0.

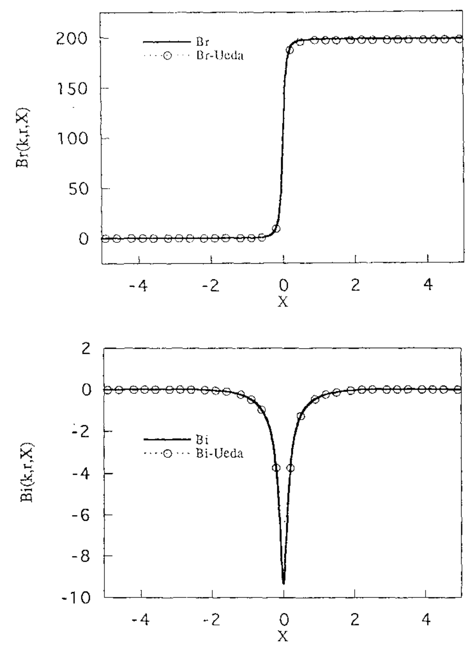

Fig. 2 Comparison between the present method and the series expansion method for k = 1.0 and r = 0.1.

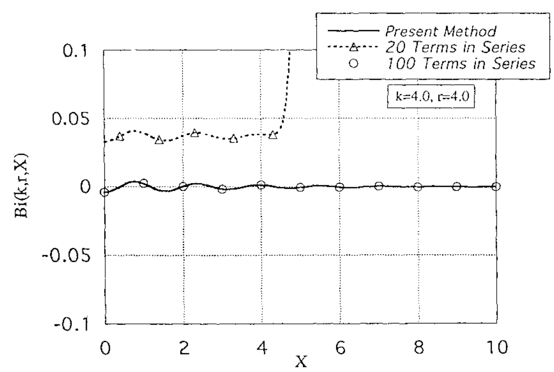

Fig. 3 Comparison between the present method and the series expansion method.

Conclusion

An alternative for the solution of the kernel function for subsonic unsteady lifting surface theory has been presented. This reduced form of the kernel function can be analytically expressed in terms of simple trigonometric and algebraic functions. Because of the closed-form nature of this solution, there is no issue of series convergence, and therefore it has a wide range of practical applicability. The results obtained compare very well to results from an alternative series expansion method. Additionally, the detailed structure and singular nature of the kernel function were clearly identified. The present solution involves employing a simple curve fit that renders the integrand readily integrable and results in an expression that is simple in form and computationally efficient.

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References

[1]Küssner, H. G., "General Airfoil Theory," NACA TR TM-979, 1941; see also *Luftfahrtforschung*, Bd. 17, Lfg. 11/12, 1940, pp. 370–378.

[2]Albano, E., and Rodden, W., "A Doublet-Lattice Method for Calculating Lift Distributions on Oscillating Surfaces in Subsonic Flows," *AIAA Journal*, Vol. 7, No. 2, 1969, pp. 279–285.

[3]Ueda, T., and Dowell, E. H., "A New Solution Method for Lifting Surfaces in Subsonic Flow," *AIAA Journal*, Vol. 20, No. 3, 1982, pp. 348–355.

[4]Eversman, W., and Pitt, D. M., "Hybrid Doublet Lattice/Doublet Point Method for Lifting Surfaces in Subsonic Flow," *AIAA Journal*, Vol. 29, No. 9, 1991, pp. 572–578.

[5]Watkins, C., Woolston, D., and Cunningham, H. J., "A Systematic Kernel Function Procedure for Determining Aerodynamic Forces on Oscillating or Steady Finite Wings at Subsonic Speeds," NACA TR R-48, 1959.

[6]Davies, D. E., "Expansions in Series of Chebyshev Polynomials of Functions Occurring in Linear Unsteady Aerodynamic Theory," British Aeronautical Research Council, TR R M 3840, 1977.

[7]Schwarz, L., "Investigation of Some Functions Related to the Cylinder Functions of Zero Order," *Luftfahrtforschung*, Vol. 20, No. 12, 1944, pp. 341–372.

[8]Desmarais, R. N., "Accurate Numerical Evaluation of Modified Struve Functions Occuring in Unsteady Aerodynamics," *Journal of Aircraft*, Vol. 16, No. 7, 1979, pp. 441–447.

[9]Laschka, B., "Zur Theorie der Harmonisch Schwingenden Tragenden Flache bei Unterschallanströmung," *Zeitschrift für Flugwissenschaften*, Vol. 11, No. 7, 1963, pp. 265–292.

[10]Dat, R., and Malfois, J. P., "Sur le Calcul du Noyau de L'Équation Integrale de la Surface Portante en Ecoulement Subsonique Instationaire," *La Recherche Aerospatiale*, No. 5, 1970, pp. 251–259.

[11]Bismarck-Nasr, M. N., "Kernel Function Occurring in Subsonic Unsteady Potential Flow," *AIAA Journal*, Vol. 29, No. 6, 1992, pp. 878, 879.

[12]Houbolt, J. C., "Some New Concepts in Oscillatory Lifting Surface Theory," Aeronautical Research Associates of Princeton, Inc., TR AFFDL-TR-69-2, 1969.

[13]Ueda, T., "Asymptotic Expansion of the Kernel Function in Subsonic Unsteady Lifting Surface Theory," *Journal of the Japan Society of Aeronautical and Space Sciences*, Vol. 29, No. 326, 1981, pp. 169–174.

Starting Point of Curvature for Reflected Diffracted Shock Wave

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Introduction

THE diffraction of a normal shock wave past a small bend was considered by Lighthill.[1] The analogous problem of a shock hitting the wall obliquely together with the reflected shock has been considered by Srivastava and Chopra.[2] Srivastava and Chopra[2] determined the pressure distribution on the wall. The present work is concerned about the start of curvature of the reflected diffracted shock wave for the case when the relative outflow behind the reflected shock before diffraction is supersonic. The corresponding work for the normal shock has been carried out by Skews.[3]

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Mathematical Formulation

In Fig. 1 the velocity, pressure, density, and sound velocity are denoted by subscripts 0 ahead of incident shock, 1 in the intermediate region, and 2 behind the reflected shock. U in the figure denotes the velocity of the point of intersection of the incident and reflected shock and δ is the angle of bend. The relations across incident and reflected shock for $\gamma = 1.4$, where γ is the ratio of specific heats, are as follows.

Across the incident shock:

$$
\begin{aligned}
q_1 &= \frac{5}{6} U \sin\alpha_0 \left(1 - \frac{a_0^2}{U^2 \sin^2\alpha_0}\right) \\
p_1 &= \frac{5}{6} \rho_0 \left(U^2 \sin^2\alpha_0 - \frac{a_0^2}{7}\right) \\
\rho_1 &= \frac{6\rho_0}{1 + \left(5a_0^2/U^2\sin^2\alpha_0\right)}, \qquad a_0 = \sqrt{\gamma p_0/\rho_0}
\end{aligned}
\tag{1a}
$$

Across the reflected shock:

$$
\begin{aligned}
\bar{q}_2 &= \bar{q}_1 + \frac{5}{6}(U^* - \bar{q}_1)\left\{1 - \frac{a_1^2}{(U^* - \bar{q}_1)^2}\right\} \\
p_2 &= \frac{5}{6}\rho_1\left\{(U^* - \bar{q}_1)^2 - (a_1^2/7)\right\} \\
\rho_2 &= \frac{6\rho_1}{\left\{1 + 5a_1^2/(U^* - \bar{q}_1)^2\right\}} \\
\bar{q}_2 &= q_2 \sin\alpha_2, \qquad U^* = U \sin\alpha_2 \\
\bar{q}_1 &= -q_1\cos(\alpha_0 + \alpha_2), \qquad a_1 = \sqrt{(\gamma p_1/\rho_1)}
\end{aligned}
\tag{1b}
$$

Srivastava and Ballabh[4] have proved that the intermediate region (region between the incident and reflected shock) would remain undisturbed for all incident shock strengths after the shock configuration has crossed the corner. This result has received experimental confirmation.[5] Let the velocity, pressure, density and entropy at any point behind the reflected diffracted shock be $\boldsymbol{q}_2'$, p_2', ρ_2', and S_2'. Choose X and Y axes with the origin at the corner and the X axis along the original wall produced. Then the equations for conservation of mass and momentum can be written as

$$\frac{\mathrm{D}\rho_2'}{\mathrm{D}t} + \rho_2' \operatorname{div} \boldsymbol{q}_2' = 0 \tag{2}$$

$$\frac{\mathrm{D}\boldsymbol{q}_2'}{\mathrm{D}t} + \frac{1}{\rho_2'}\nabla p_2' = 0 \tag{3}$$

If there is no heat conduction or radiation, the entropy, satisfies the equation $\mathrm{D}S_2'/\mathrm{D}t = 0$. Using Lighthill's linearized theory[1] and the transformations

$$
\begin{aligned}
x &= \frac{X - q_2 t}{a_2 t}, \qquad y = \frac{Y}{a_2 t} \\
p &= \frac{p_2' - p_2}{a_2\rho_2 q_2}, \qquad \frac{\boldsymbol{q}_2'}{q_2} = (1 + u, v)
\end{aligned}
\tag{4}
$$

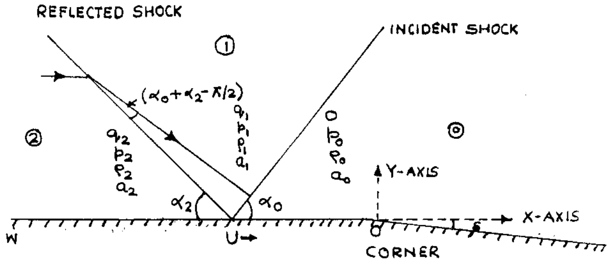

Fig. 1 Schematic drawing for oblique shock configuration passing over a small bend.